5.5b local stability of first order systems

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Recall: We can a Taylor's Thm to approximate a nonlinear system.

Thm: Let $\dot{X}(t) = F(X(t))$ be a system of autonomous 1st-order ODEs

(viz Vid 2.8)

$X(t) = (x_1(t), \dots, x_n(t))^T$, $F = (f_1, \dots, f_n)^T$, $f_i = f_i(x_1, \dots, x_n)$.

Let $\bar{X}$ be an equilibrium of the system. Then the linearization of the system about $\bar{X}$ and letting $U(t) = X(t) - \bar{X}$ gives a system

$$\dot{U}(t) = J\,U(t),$$

where J is the Jacobian matrix of F at $\bar{X}$,

$$J(\bar{X}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \left.\vphantom{\begin{pmatrix}a\\a\\a\end{pmatrix}}\right\} \text{Assume all partial derivatives are continuous in an open neighborhood of } \bar{X}$$

Then $\bar{X}$ is locally asymp stable if $Re(\lambda_i) < 0$ $\forall$ eigenvalues λ_i and unstable if some $Re(\lambda_i) > 0$.

proof. sketch

$$\dot{X}(t) = F(X) \approx \underbrace{F(\bar{X})}_{0} + \underbrace{J(\bar{x})}_{\text{Jacobian}}(X(t) - \bar{x}) + \cancel{\tfrac{1}{2}(X(t)-\bar{x})^T \underbrace{H(\bar{x})}_{\text{Hessian}}(X(t)-\bar{x}) + \cdots}$$

$$\Rightarrow \dot{X}(t) \approx J(\bar{x})(X(t) - \bar{x}) \quad \text{for } X(t) \text{ sufficiently close to } \bar{X}$$

$$\dot{U}(t) = J(\bar{x})U(t)$$

$$U(t) = e^{tJ(\bar{x})}U(0).$$

Let $PBP^{-1} = J(\bar{x})$, where B is in Jordan canonical form, $B = \Lambda + N$, where Λ is a diagonal matrix, and N only has nonzero entries directly above the diagonal (some of which are 1, corresponding to Jordan blocks, and some of which are 0).

$$\begin{bmatrix} \lambda_1 & 1 & & & \\ & \lambda_1 & 0 & & \\ & & \lambda_2 & 0 & \\ & & & \lambda_3 & \ddots \\ & & & & \ddots \end{bmatrix}$$

Then $e^{tJ(\bar{x})} = P\exp(t\Lambda + tN)P^{-1}$
$= P\exp(t\Lambda)\exp(tN)P^{-1}$ $N^n = 0$

Note that N is nilpotent, and so is tN.
Thus, the power series of $\exp(tN)$ can be cut after n terms, so
$$\exp(tN) = \sum_{m=0}^{n-1} t^m C_m, \quad \text{where} \quad C_m \in \mathbb{C}^{n\times n} \text{ has no dependence on } t.$$
$C_m = \frac{1}{m!}N^m$.

$$\begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots \\ \vdots & \ddots & \\ & & p_{nn}(t) \end{bmatrix}, \quad p_{ij}(t) \text{ are degree } n-1 \text{ polynomials}$$

Also, $\exp(t\Lambda)$ is a diagonal matrix with terms $e^{\lambda_i t}$.
But $\lim_{t\to\infty} e^{-rt}\cdot t^m = 0$ for any $r > 0$ and integer $m < n$.

Thus, $\lim_{t\to\infty} \exp(t\Lambda)\exp(tN) = 0$.
$\Rightarrow \lim_{t\to\infty} U(t) = \lim_{t\to\infty} \exp(tJ(\bar{x}))U_0 = 0$.
$\Rightarrow \lim_{t\to\infty} x(t) = \bar{x}$, so locally asymptotically stable.

If any $\mathrm{Re}(\lambda_i) > 0$, then $\lim_{t\to\infty} U(t) = \infty$, if $U(0) = V_i$, the associated eigenvector.
Of course, the linearization may break down as $|x(t) - \bar{x}|$ grows, but $x(t)$ will still leave a sufficiently small ball around $\bar{x}$, so it is unstable. $\square$

Def. 5.6 Let $\dot{x} = F(x)$ have equilibrium $\bar{x}$, and let J be the Jacobian of F at $\bar{x}$. Then $\bar{x}$ is **hyperbolic** if all eigenvalues of J have nonzero real part and **nonhyperbolic** otherwise.

Thm 5.4 Let $\dot{x} = F(x)$, $x \in \mathbb{R}^2$, and the partial derivatives of F are continuous in an open neighborhood of an equilibrium $\bar{x}$. Then $\bar{x}$ is locally asymptotically stable if
$$T_r(J) < 0 \quad \text{and} \quad \det(J) > 0,$$

where J is the Jacobian at $\bar{X}$. It is unstable if either $Tr(J) > 0$ or $det(J) < 0$.

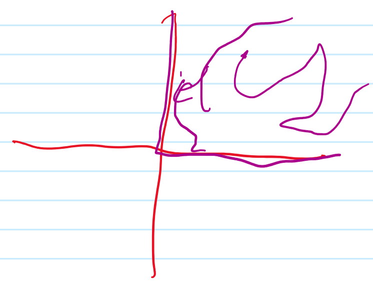
Ex. 5.11 Consider a predator-prey model where

$x(t)$ = density of prey species

$y(t)$ = density of predator species

$$\frac{dx}{dt} = x\big(\underbrace{\underset{\uparrow \text{ growth rate}}{r} - r\cdot\frac{x}{\underset{\uparrow\text{ carrying capacity}}{K}}}_{\text{logistic growth}} - \underbrace{ay}_{\text{predator}}\big), \quad r, K, a > 0$$

$$\frac{dy}{dt} = \underbrace{y(-b}_{\text{death rate}} + \underbrace{cx}_{\text{growth as result of eating prey.}}), \quad b, c > 0$$



Note: If $x(0) = 0$, then $x(t) = 0 \;\; \forall\, t \geq 0$

If $y(0) = 0$, then $y(t) = 0 \;\; \forall\, t \geq 0$

Thus, because trajectories cannot cross, positive solutions cannot cross either axis. Thus, the positive quadrant $x \geq 0$, $y \geq 0$ is **positively invariant**. i.e., if $(x(0), y(0)) \in \mathbb{R}_+^2$, then $(x(t), y(t)) \in \mathbb{R}_+^2 \;\; \forall\, t \geq 0$.

3 equilibria: $\underbrace{(0,0)}_{\text{no predators or prey}}$, $\underbrace{(K,0)}_{\text{only prey, no predators}}$, and $(\bar{x}, \bar{y}) = \Big(\frac{b}{c}, \underbrace{\frac{r(cK-b)}{acK}}_{\text{positive only if } K > \frac{b}{c}.}\Big)$

If neg, prey population not large enough to support predators.

$$\frac{dx}{dt} = x\left(r - r\frac{x}{K} - ay\right)$$

$$\frac{dy}{dt} = y(-b + cx)$$

$$J(x,y) = \begin{pmatrix} r - \frac{2rx}{K} - ay & -ax \\ cy & -b + cx \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} r & 0 \\ 0 & -b \end{pmatrix}$$

$\lambda_1 = r, \quad \lambda_2 = -b$

$\Rightarrow$ saddle pt / unstable

$$J(K,0) = \begin{pmatrix} -r & -aK \\ 0 & -b + cK \end{pmatrix}$$

$\lambda_1 = -r \quad \lambda_2 = -b + cK$

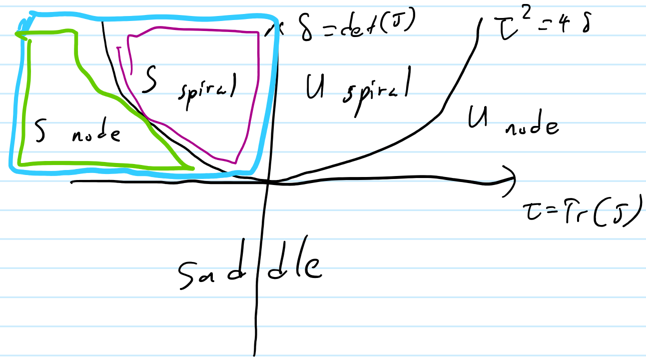
If $K > \frac{b}{c}$, then $\lambda_2 > 0$, so saddle pt / unstable

$\Rightarrow$ saddle pt / unstable

If $K > \frac{b}{c}$, then $\lambda_2 > 0$, so saddle pt / unstable

If $K < \frac{b}{c}$, then $\lambda_2 < 0$, so stable node.

$$\bar{x} = \frac{b}{c} \qquad \bar{y} = \frac{r(cK-b)}{acK}$$



$$J(\bar{x},\bar{y}) = \begin{pmatrix} -\frac{rb}{Kc} & -\frac{ab}{c} \\ \frac{r(cK-b)}{aK} & 0 \end{pmatrix}$$

$$Tr(J) = \frac{-rb}{Kc} < 0$$

$$\det(J) = \frac{ab}{c} \cdot \frac{r(cK-b)}{aK} > 0 \quad \text{if} \quad K > \frac{b}{c}$$

$\Big\}$ $(\bar{x}, \bar{y})$ is locally asymptotically stable if $K > \frac{b}{c}$.

discriminant is $\gamma = Tr(J)^2 - 4\det(J) = \left[-\frac{rb}{cK}\right]^2 - 4 \cdot \frac{rb(cK-b)}{cK}$.

If $\gamma > 0$, then node.

If $\gamma < 0$, then spiral.